5.6 COMPLEX ZEROS OF POLYNOMIAL FUNCTIONS; FUNDAMENTAL THEOREM OF ALGEBRA

Def.: A complex variable is a variable in complex number system where the complex polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ has $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ complex numbers; n is a none negative ~~p~~ integer $n \in \{0, 1, 2, 3, \ldots\}$ a_n is the leading coefficient, then there is a complex number r such that r is a complex zero of $f(x)$, if $f(r) = 0$

FUNDAMENTAL THEOREM OF ALGEBRA

Every complex function with degree $n \geq 1$ has at least ~~a~~ one complex zero.

Every complex function can be factored under the complex number system as

$$f(x) = a_n (x - r_1)(x - r_2) \ldots (x - r_{n-1})(x - r_n)$$

as linear factors ($(x - r)$ is linear)

THEOREM: if $r = a+bi$ is a complex zero of a polynomial function $f(x)$, then the conjugate $r = a-bi$ is also a zero of $f(x)$.

Ex: 10/394. Degree 4; ZEROS 1, 2, 2+i

$r_1 = 1$

$r_2 = 2$

$r_3 = 2+i$

$r_4 = 2-i$

Ex: 16/394. Degree 6; zeros: i, $3-2i$, $-2+i$

$r_1 = i$ $\quad$ $r_4 = -i$

$r_2 = 3-2i$ $\quad$ $r_5 = 3+2i$

$r_3 = -2+i$ $\quad$ $r_6 = -2-i$

Ex: 18/394. Degree 4; ZEROS: i, $1-2i$

$r_1 = i$ $\quad$ $r_3 = -i$

$r_2 = 1-2i$ $\quad$ $r_4 = 1+2i$

$$f(x) = 1(x-r_1)(x-r_2)(x-r_3)(x-r_4)$$

$$= 1(x-i)(x-(1-2i))(x-(-i))(x-(1+2i))$$

+

$= (x-i)(x+i)(x-1+2i)(x-1-2i)$

$= (x^2 + \cancel{xi} - \cancel{xi} - i^2)(x^2 - x - \cancel{2xi} - x + 1 + \cancel{2i} + \cancel{2xi} - \cancel{2i} - 4i^2)$

$= (x^2+1)(x^2-2x+1+4)$

$= (x^2+1)(x^2-2x+5)$

$= x^4 - 2x^3 + \underbrace{5x^2 + x^2} - 2x + 5$

$= x^4 - 2x^3 + 6x^2 - 2x + 5$

Ex 22/394 Degree 5; zeros: 1 multiplicity 3

$a_n = 1.$ $\qquad 1+i \Rightarrow 1-i$

$f(x) = 1(x-1)^3(x-(1+i))(x-(1-i))$

$= (x-1)^3(x-1-i)(x-1+i)$

$= (x^3 - 3x^2 + 3x - 1)(x^2 - x + \cancel{xi} - x + 1 - \cancel{i} - \cancel{xi} + \cancel{i} - i^2)$
$\qquad\qquad +1$

$= (x^3 - 3x^2 + 3x - 1)(x^2 - 2x + 2)$

$= x^5 - 2x^4 + 2x^3 - 3x^4 + 6x^3 - 6x^2 + 3x^3 - 6x^2 + 6x - x^2 + 2x - 2$

$= x^5 - 5x^4 + 11x^3 - 13x^2 + 8x - 2$

THEOREM

A polynomial function with real coefficients can be factored in real number system as a product of linear ~~fc~~ factors or irreducible quadratic

Ex: $f(x) = (x-3)(x+\frac{1}{2})(x^2+1)$

$(x-3)$ ↓ linear, $(x+\frac{1}{2})$ ↓ linear, (x^2+1) ↓ irreducible quadratic

Ex: 24/394.

$g(x) = x^3+3x^2+25x+75$ zero: $-5i \Rightarrow 5i$

$(x-5i)(x+5i) = x^2+\cancel{5xi}-\cancel{5xi}-25i^2$

$= x^2+25.$

$g(x) = (x^2+25)(\quad ? \quad)$

$$\begin{array}{r}
x+3 \\
x^2+25 \,\overline{)\, x^3+3x^2+25x+75} \\
-(x^3+0x^2+25x) \\
\hline
3x^2 +75 \\
-(3x^2 +75) \\
\hline
0
\end{array}$$

$x(x^2+25)$

$= x^3+25x$

$3(x^2+25)$

$= 3x^2+75$

$g(x) = (x^2+25)(x+3)$

zeros: $-5i$

$5i$

$-3.$

$g(x) = \underbrace{x^3 + 3x^2} + \underbrace{25x + 75}$

$= x^2(x+3) + 25(x+3)$

$= (x^2+25)(x+3)$

$\downarrow \qquad \downarrow$

$+5i \qquad -3$

$-5i$

Ex 32/395 $f(x) = x^4 - 1$

$f(x) = (x^2-1)(x^2+1)$

$= (x+1)(x-1)(x^2+1)$

$(x+1)(x-1)(x^2+1) = 0$

$x+1=0 \Rightarrow x = -1$

$x-1=0 \Rightarrow x = 1$

$x^2+1=0 \Rightarrow \sqrt{x^2} = \sqrt{-1}$

$x = \pm\sqrt{-1}$

$x = \pm i$

$x = -1$

$x = 1$

$x = i$

$x = -i$

Ex: 36/395 $\quad f(x) = x^4 + 13x^2 + 36$

$$= (x^2+9)(x^2+4)$$

$$(x^2+9)(x^2+4) = 0$$

$$x^2+9=0 \Rightarrow \sqrt{x^2} = \sqrt{-9}$$

$$x = \pm 3i$$

$$x^2+4=0 \Rightarrow \sqrt{x^2} = \sqrt{-4}$$

$$x = \pm 2i$$

$$x = 3i, -3i, 2i, -2i$$

Ex40: $f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65$

$$\frac{p}{q} = \frac{\pm 1, \pm 5, \pm 13, \pm 65}{\pm 1, \pm 2} = \pm 1; \pm 5; \pm 13; \pm 65; \pm \frac{1}{2}$$

$$\pm \frac{5}{2}, \pm \frac{13}{2}; \pm \frac{65}{2}$$

$x = 5 \Rightarrow (x-5)(2x^3 + 11x^2 + 20x - 13)$

5	2	1	-35	-113	65
		10	55	100	-65
	2	11	20	-13	$\boxed{0}$

$$2x^3+11x^2+20x-13$$

$x=\frac{1}{2}$ $\quad \cancel{2}\cdot\frac{1}{\cancel{8}_4}+11\cdot\frac{1}{4}+\cancel{20}^{10}\cdot\frac{1}{\cancel{2}}-13$

$$=\frac{1}{4}+\frac{11}{4}+10-13 = \frac{12}{4}-3 = 3-3=0.$$

$$\begin{array}{c|cccc}
\frac{1}{2} & 2 & 11 & 20 & -13 \\
 & \downarrow & 1 & 6 & 13 \\
\hline
 & 2 & 12 & 26 & \boxed{0}
\end{array}$$

$$f(x)=(x-5)\left(x-\frac{1}{2}\right)\left(2x^2+12x+26\right)$$

$$2x^2+12x+26 = 2(x^2+6x+13)$$

$$x=\frac{-6\pm\sqrt{36-52}}{2}=\frac{-6\pm\sqrt{-16}}{2}$$

$$=-\frac{6}{2}\pm\frac{4i}{2}= \begin{cases}-3+2i\\ -3-2i\end{cases}$$

$$f(x)=2(x-5)\left(x-\frac{1}{2}\right)(x+3-2i)(x+3+2i)$$