Rational Exponents and Radical Expressions

A. Simplify rational exponents

If n is a positive integer, then $a^{\frac{1}{n}}$ is a number, whose nTh power is a.

Example:

$144^{1/2} = 12, \quad 12^2 = 144$ $\qquad 144^{\frac{1}{2}} =$

$8^{1/3} = 2, \quad 2^3 = 8$

$16 \cdot 16 = 96 + 160 = 256 \qquad 3$

$2 \cdot 2 \cdot 2 \cdot 2 \Rightarrow 2^4 = 16$

$\underbrace{2 \cdot 2}_{4} \cdot \underbrace{2 \cdot 2}_{4}$, $4 \cdot 4 = 16$

$3 \cdot 3 \cdot 3 \cdot 3 \Rightarrow 3^4 = 81$

$\underbrace{3 \cdot 3}_{9} \cdot \underbrace{3 \cdot 3}_{9}$, $9 \cdot 9 = 81$

$4 \cdot 4 \cdot 4 \cdot 4 = 256 \Rightarrow 4^4 = 256$

$\underbrace{4 \cdot 4}_{16} \cdot \underbrace{4 \cdot 4}_{16}$

Exercise 1:

a) Simplify: $125^{1/3} = 5$

b) Simplify: $256^{\frac{1}{4}} = 4$

In m and n are positive integers and $a^{\frac{1}{n}}$ is a real number, then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$$

Example:

$27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = 3^2 = 9, \qquad 27^{\frac{1}{3}} = 3 \quad 3^3 = 27$

$32^{\frac{-3}{5}} = \left(32^{\frac{1}{5}}\right)^{-3} = 2^{-3} = \frac{1}{8},$

$?^5 = 32 \Rightarrow 2^5 = 32$

$64^{\frac{2}{3}} = \left(64^{\frac{1}{3}}\right)^2 \qquad 4^3 = 64$

$= 4^2 = 16$

Exercise 2:

a) Simplify: $121^{3/2} = 11^3 = \boxed{1331}$

b) Simplify: $\left(\frac{9}{16}\right)^{-3/2} = \left(\frac{9^{\frac{1}{2}}}{16^{\frac{1}{2}}}\right)^{-3} = \left(\frac{3}{4}\right)^{-3} = \left(\frac{3^{-3}}{4^{-3}}\right) = \frac{4^3}{3^3} = \frac{64}{27}$

Exercise 3:

a) Simplify: $a^{1/3}a^{20/3} = a^{\frac{1}{3}+\frac{20}{3}} = a^{\frac{21}{3}} = a^{7}$

b) Simplify: $\frac{a^{-2/5}}{a^{3/4}} = a^{-\frac{2}{5}-\frac{3}{4}} = a^{-\frac{2\cdot4}{5\cdot4}-\frac{3\cdot5}{4\cdot5}} = a^{-\frac{8}{20}-\frac{15}{20}}$

$= a^{-\frac{23}{20}} = \frac{1}{a^{23/20}}$

c) Simplify: $\left(x^{-9/8}\right)^{72}$

$x^{\left(-\frac{9}{\cancel{8}}\right)\left(\overset{9}{\cancel{72}}\right)} = x^{-81} = \frac{1}{x^{81}}$

B. Write exponential expressions as radical expressions and radical expressions as exponential expressions

If a is a real number and n is a positive integer, then $a^{\frac{1}{n}} = \sqrt[n]{a}$

INDEX OF THE RADICAL $\leftarrow \sqrt[n]{a} \rightarrow$ radical

RADICAND

Special radicals

$a^{\frac{1}{2}} = \sqrt[2]{a} = \sqrt{a}$ square root.

$a^{\frac{1}{5}} = \sqrt[5]{a}$ 5th root

$a^{\frac{1}{3}} = \sqrt[3]{a}$ cube root

$a^{\frac{1}{4}} = \sqrt[4]{a}$ 4th root

Square Roots		*Cube Roots*	*Fourth Roots*	*Fifth Roots*
$\sqrt{1} = 1$	$\sqrt{36} = 6$	$\sqrt[3]{1} = 1$	$\sqrt[4]{1} = 1$	$\sqrt[5]{1} = 1$
$\sqrt{4} = 2$	$\sqrt{49} = 7$	$\sqrt[3]{8} = 2$	$\sqrt[4]{16} = 2$	$\sqrt[5]{32} = 2$
$\sqrt{9} = 3$	$\sqrt{64} = 8$	$\sqrt[3]{27} = 3$	$\sqrt[4]{81} = 3$	$\sqrt[5]{243} = 3$
$\sqrt{16} = 4$	$\sqrt{81} = 9$	$\sqrt[3]{64} = 4$	$\sqrt[4]{256} = 4$	
$\sqrt{25} = 5$	$\sqrt{100} = 10$	$\sqrt[3]{125} = 5$	$\sqrt[4]{625} = 5$	
$\sqrt{121} = 11$	$\sqrt{144} = 12$			
$\sqrt{169} = 13$	$\sqrt{196} = 14$			

Example:

$65^{1/3} = \sqrt[3]{65}$ $\quad 5^{\frac{1}{3}} = \sqrt[3]{5}$ $\quad 16^{\frac{1}{5}} = \sqrt[5]{16}$

$\sqrt{6} = 6^{\frac{1}{2}}$ $\quad \sqrt[3]{15} = 15^{\frac{1}{3}}$ $\quad \sqrt[10]{8} = 8^{\frac{1}{10}}$

Exercise 4: Write as a radical

a) $9^{1/4} =$ $\sqrt[4]{9}$

b) $25^{\frac{1}{10}} =$ $\sqrt[10]{25}$

Exercise 5: Write as an exponential expression

a) $\sqrt[5]{101} =$ $101^{1/5}$

b) $\sqrt{12} =$ $12^{\frac{1}{2}}$

If a is a real number and n and m are a positive integer, then

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Example: Rewrite the following exponential expression as a radical expression:

a) $a^{2/5} = \sqrt[5]{a^2}$ or $(\sqrt[5]{a})^2$

b) $(5v)^{1/2} = \sqrt{5v}$

$5v^{\frac{1}{2}} = 5\sqrt{v}$

$(4x)^{3/9} = \sqrt[9]{(4x)^3} = \sqrt[9]{64x^3}$

Exercise 6 Rewrite the following exponential expression as a radical expression:

a) $-8x^{8/9}$ $-8\sqrt[9]{x^8}$ $(-8x)^{8/9} = \sqrt[9]{(-8x)^8}$

b) $x^{-8/3}$ $\sqrt[3]{x^{-8}} = \sqrt[3]{\frac{1}{x^8}}$

c) $(a^5b)^{\frac{5}{6}}$ $\sqrt[6]{(a^5b)^5}$

If a number is not a perfect power, its root can only be approximated; examples include $\sqrt{2}$ and $\sqrt[3]{5}$. These numbers are **irrational numbers**.

Example of irrational numbers: $\sqrt{5} = 2.2360679\ldots$ $\sqrt[3]{3} = 1.4422495\ldots$

! IF THE INDEX OF THE RADICAL IS AN EVEN NUMBER THEN THE RADICAND MUST BE POSITIVE. IF THE RADICAND IS NEGATIVE THEN IS NOT REAL NUMBER

C. Simplify radical expressions

Example: Simplify:

a) $-\sqrt{x^{12}} = -(x^{12})^{\frac{1}{2}} = -x^{12\cdot\frac{1}{2}} = -x^{6}$

b) $\sqrt[3]{-x^6y^{27}} = (-x^6y^{27})^{\frac{1}{3}} = -x^{6\cdot\frac{1}{3}}y^{27\cdot\frac{1}{3}} = -x^{2}y^{9}$

c) $\sqrt[3]{-64x^{18}y^{15}} = -4^{3/3}x^{\frac{18}{3}}y^{\frac{15}{3}} = -4x^{6}y^{5}$

d) $\sqrt{-25x^2y^4} =$ NOT REAL NUMBER

$?^2 = -25$

$5^2 = 25$

$(-5)^2 = 25$

$\sqrt{-9}$

$3^2 = 9$

$(-3)^2 = 9$

Exercise 7: Simplify

a) $-\sqrt[5]{-x^{30}y^{10}} = -(-x^{30}y^{10})^{\frac{1}{5}} = x^6y^2$

b) $-\sqrt[4]{x^{24}y^{16}} = -(x^{24}y^{16})^{\frac{1}{4}} = -x^6y^4$

c) $\sqrt[3]{343x^{24}} = 343^{\frac{1}{3}}x^{24\cdot\frac{1}{3}} = 7x^{8}$

$7^{3\cdot\frac{1}{3}}$

d) $\sqrt[4]{16x^{20}y^{24}} = (2^4x^{20}y^{24})^{\frac{1}{4}} = 2x^5y^6$

D. Simplify ~~rational~~ radical expression that are not perfect roots:

A radical expression is not in simplest form if the radicand contains a factor that is a perfect power of the index.

Example: Simplify

$\sqrt{32} =$ $\qquad$ $32 = 16 \cdot 2 = 4^2 \cdot 2$ $\qquad$ $\sqrt{32} = 4\sqrt{2}$

<u>*Product property of radicals*</u>

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$.

This property works for $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Example: Simplify

a) $\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$

b) $-\sqrt[3]{125x^{17}y^7} = -\sqrt[3]{5^3 x^{15} \cdot x^2 y^6 y} = -\sqrt[3]{5^3 x^{15} y^6}\ \sqrt[3]{x^2 y}$
$= -5x^5 y^2 \sqrt[3]{x^2 y}$

c) $\sqrt[3]{a^{28}b^{20}} = \sqrt[3]{a^{27}a \cdot b^{18}b^2} = \sqrt[3]{a^{27}b^{18}}\ \sqrt[3]{ab^2} = a^9 b^6 \sqrt[3]{ab^2}$

Exercise 8: Simplify

a) $\sqrt{72a^{15}b^3c} = \sqrt{8 \cdot 9a^{15}b^3c} = \sqrt{4 \cdot 2 \cdot 9\, a^{14} a\, b^2 b\, c} = \sqrt{4 \cdot 9\, a^{14} b^2}\ \sqrt{2abc}$
$= 2 \cdot 3a^7 b \sqrt{2abc}$
$= 6a^7 b \sqrt{2abc}$

b) $\sqrt{20x} = \sqrt{4 \cdot 5 \cdot x} = \sqrt{4}\ \sqrt{5x} = \boxed{2\sqrt{5x}}$

c) $\sqrt[3]{128} = \sqrt[3]{2^7} = \sqrt[3]{2^6 2^1} = \sqrt[3]{2^6}\ \sqrt[3]{2^1} = 2^2 \sqrt[3]{2} = \boxed{4\sqrt[3]{2}}$

d) $\sqrt[3]{16} = \sqrt[3]{2^4} = \sqrt[3]{2^3}\ \sqrt[3]{2} = \boxed{2\sqrt[3]{2}}$

e) $8a\sqrt[4]{16ab^5} = 8a \cdot 2b\sqrt[4]{ab} = 16ab\sqrt[4]{ab}$

$\sqrt[4]{16ab^5} = \sqrt[4]{2^4 a b^5} = \sqrt[4]{\underline{2^4} a \underline{b^4} \cdot b}$

$= \sqrt[4]{2^4 b^4} \quad \sqrt[4]{ab}$

$= 2b\sqrt[4]{ab}$

$\sqrt{72} = \sqrt{8 \cdot 9} = \sqrt{4 \cdot 9 \cdot 2} = \sqrt{4 \cdot 9}\ \sqrt{2}$

$4 \cdot 2$

$= 2 \cdot 3\sqrt{2} = 6\sqrt{2}$

$\sqrt{72} = \sqrt{\underline{36} \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}$

$\sqrt{72} = \sqrt{2^3 \cdot 3^2} = \sqrt{3^2 \cdot 2^2}\ \sqrt{2}$

$2^2 \cdot 2$

$= 3 \cdot 2\sqrt{2}$

$= 6\sqrt{2}$

72

2    $72 \div 2 = 36$

2    $36 \div 2 = 18$

2   9

$3 \cdot 3$