PROPERTIES OF LOGARITHMS.

1. $\boxed{\log_a 1 = 0}$

2. $\boxed{\log_a a = 1}$

3. $\boxed{\log_a a^x = x}$ 4. $\boxed{a^{\log_a x} = x}$

5. $\boxed{\log_a u = \log_a v \iff u = v}$

The Product Property of Logarithms.

$$\boxed{\log_a (M \cdot N) = \log_a M + \log_a N.}$$

<u>Proof</u>: $A = \log_a M \Rightarrow M = a^A$

$B = \log_a N \Rightarrow N = a^B$

$$\underline{\log_a (M \cdot N)} = \log_a (a^A \cdot a^B) = \log_a a^{A+B} = A+B$$

$$= \underline{\log_a M + \log_a N}$$

Ex: $\log_2 x \cdot y = \log_2 x + \log_2 y$

$\log_3 (x+2)(x-5) = \log_3 (x+2) + \log_3 (x-5)$

The Quotient Property of Logarithms

$$\boxed{\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N.}$$

Proof: $A = \log_a M \Rightarrow M = a^A$

$B = \log_a N \Rightarrow N = a^B$

$\underline{\log_a \left(\frac{M}{N}\right)} = \log_a \left(\frac{a^A}{a^B}\right) = \log_a a^{A-B} = A - B$

$= \underline{\log_a M - \log_a N}$

Ex: $\log_2 \frac{x}{y} = \log_2 x - \log_2 y$

$\log_5 \frac{x-3}{x+2} = \log_5 (x-3) - \log_5 (x+2)$

The Exponential Property of Logarithms

$$\boxed{\log_a M^r = r \cdot \log_a M}$$

Proof $A = \log_a M \Rightarrow M = a^A$.

$$\underline{\log_a M^r} = \log_a (a^A)^r = \log_a a^{rA} = rA$$

$$= \underline{r \log_a M}.$$

$$\boxed{a^x = e^{x \ln a}}$$

Proof.

$$a^{\log_a x} = x$$

$$e^{\log_e e} = e$$

$$a^x = e \quad (1)$$

$$\ln a^x = \ln e$$

$$x \ln a = \ln e$$

$$e^{x \ln a} = e^{\ln e}$$

$$e^{x \ln a} = e \quad (2)$$

$$(1),(2) \Rightarrow \underline{a^x = e^{x \ln a}}$$

Ex: $\log_3 x^3 = 3\log_3 x$

$$\log_2 \sqrt{x-2} = \log_2 (x-2)^{\frac{1}{2}} = \frac{1}{2}\log_2 (x-2)$$

Ex: $\log_5 \left(\dfrac{\sqrt[3]{x^2+1}}{x^2-1} \right) = \log_5 (x^2+1)^{\frac{1}{3}} - \log_5 (x^2-1)$

$$= \frac{1}{3}\log_5 (x^2+1) - \log_5 (x^2-1)$$

Ex: $\log_4 (x^2-1) - 5\log_4 (x+1)$

$$= \log_4 (x^2-1) - \log_4 (x+1)^5$$

$$= \log_4 \frac{x^2-1}{(x+1)^5} = \log_4 \frac{\cancel{(x+1)}(x-1)}{(x+1)^{\cancel{5}\,4}}$$

$$= \log_4 \frac{x-1}{(x+1)^4}$$

ONE - TO - ONE PROPERTY

$$\log_a M = \log_a N \iff M = N$$

Ex: $\log_a (x+2) = \log_a (x-1) \Rightarrow x+2 = x-1$

CHANGING BASE FORMULA

$$\log_a M = \frac{\log M}{\log a} = \frac{\ln M}{\ln a} = \frac{\log_b M}{\log_b a}$$

Ex: $\log_2 3 = \dfrac{\log 3}{\log 2} \approx 1.585$

$$\log_\pi 52 = \frac{\ln 52}{\ln \pi} \approx 3.452$$

Ex: $\ln\left[\frac{5x^2\sqrt[3]{1-x}}{4(x+1)^2}\right] = \ln\left(5x^2\sqrt[3]{1-x}\right) - \ln\left[4(x+1)^2\right]$

$$= \ln 5 + \ln x^2 + \ln(1-x)^{\frac{1}{3}} - \left(\ln 4 + \ln(x+1)^2\right)$$

$$= \ln 5 + 2\ln x + \frac{1}{3}\ln(1-x) - \ln 4 - 2\ln(x+1)$$

Ex: $\log\left[\frac{x(x+5)}{x^2-2x+1}\right]^2$

$$= 2\log x(x+5) - 2\log(x^2-2x+1)$$

$$\left.\begin{aligned} &= 2\log x + 2\log(x+5) - 2\log(x^2-2x+1) \\ &= 2\log x + 2\log(x+5) - 2\log(x^2-2x+1) \end{aligned}\right\} \text{Same}$$

$$= 2\log x + 2\log(x+5) - 2\log\underbrace{(x-1)(x-1)}_{(x-1)^2}$$

$$= 2\log x + 2\log(x+5) - \left(2\log(x-1) + 2\log(x-1)\right)$$

$$= 2\log x + 2\log(x+5) - 2\log(x-1) - 2\log(x-1)$$

$$= 2\log x + 2\log(x+5) - 4\log(x-1)$$

Ex: $3\log_5(3x+1) - 2\log_5(2x-1) - \log_5 x$

$= \log_5(3x+1)^3 - \log_5(2x-1)^2 - \log_5 x$

$= \log_5 \dfrac{(3x+1)^3}{(2x-1)^2} - \log_5 x$

$= \log_5 \dfrac{\frac{(3x+1)^3}{(2x-1)^2}}{\frac{x}{1}}$

$= \log_5 \dfrac{(3x+1)^3}{(2x-1)^2} \cdot \dfrac{1}{x} = \log_5 \dfrac{(3x+1)^3}{x(2x-1)^2}$

Ex: $\frac{1}{3}\log(x^3+1) + \frac{1}{2}\log(x^2+1)$

$= \log(x^3+1)^{\frac{1}{3}} + \log(x^2+1)^{\frac{1}{2}}$

$= \log\left(\sqrt[3]{x^3+1}\right) + \log\left(\sqrt{x^2+1}\right)$

$= \log\left(\sqrt[3]{x^3+1}\right)\left(\sqrt{x^2+1}\right)$