LOGARITHMIC FUNCTIONS

Definition

For $x > 0$ and $0 < a \neq 1$,

$y = \log_a x$ if and only if $x = a^y$.

The function given by $f(x) = \log_a x$ is called the **logarithmic function with base a.**

Every logarithmic equation has an equivalent exponential form:

$y = \log_a x$ is equivalent to $x = a^y$.

A logarithm is an exponent!

A logarithmic function is the inverse function of an exponential function.

Exponential function: $y = a^x$

Logarithmic function: $y = \log_a x$ is equivalent to $x = a^y$

Examples

Examples: Write the equivalent exponential equation and solve for y.

Logarithmic Equation	**Equivalent Exponential Equation**	**Solution**
$y = \log_3 9$	$3^y = 9$	$3^y = 3^2 \Rightarrow y = 2$
$y = \log_2 8$	$2^y = 8$	$2^y = 2^3 \Rightarrow y = 3$
$y = \log_{\frac{1}{2}} 1$	$\frac{1}{2}^y = 1$ *or* $2^{-y} = 1$	$2^{-y} = 2^0 \Rightarrow y = 0$
$y = \log_4 \frac{1}{16}$	$4^y = \frac{1}{16}$	$4^y = 4^{-2} \Rightarrow y = -2$

The base 10 logarithm function $f(x) = \log_{10} x$ is called the common logarithm function.

The LOG key on a calculator is used to obtain common logarithms.

Examples: Calculate the values using a calculator.

Function Value	Keystrokes	Display
$\log_{10} 100$	LOG 100 ENTER	2
$\log_{10}(\frac{2}{5})$	LOG (2 ÷ 5) ENTER	– 0.3979400
$\log_{10} 5$	LOG 5 ENTER	0.6989700
$\log_{10} -4$	LOG –4 ENTER	ERROR

no power of 10 gives a negative number

Change Exponentials Expressions into Logarithmic Expressions

$e^{2.2} = M$ $\qquad \log_e M = 2.2$

$4^x = 1.8$ $\qquad \log_4 1.8 = x$

$10^x = 2$ $\qquad \log_{10} 2 = x$

$y = \log_a x \Rightarrow a^y = x$

Properties of Logarithms

Properties of Logarithms

1. $\log_a 1 = 0$ since $a^0 = 1$.
2. $\log_a a = 1$ since $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ inverse property
4. If $\log_a x = \log_a y$, then $x = y$. one-to-one property

Examples: Solve for x: $\log_3 3 = x$

$\log_3 3 = 1$ property 2 $\rightarrow$ $x = 1$

Simplify: $\log_4 4^5$

$\log_4 4^5 = 5$ property 3

Simplify: $8^{\log_8 12}$

$8^{\log_8 12} = 12$ property 3

$y = \log_a 1 \qquad a^y = 1 \Rightarrow a^0 = 1$

$\Rightarrow y = 0$

$\log_2 2^9 = 9$

$e^{\log_e 4} = 4$

$$f \circ f^{-1} = x \qquad f^{-1} \circ f = x$$

$$y = \log_a x \Rightarrow f(x) = \log_a x$$

$$y = a^x \qquad f^{-1}(x) = a^x$$

$$f \circ f^{-1} = \log_a a^x = x$$

$$f^{-1} \circ f = a^{\log_a x} = x$$

composition of inverse functions

DOMAIN

$$l(x) = \log_2 (x^2 - 4)$$

$x^2 - 4 > 0$

$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = -2 \quad x = 2$

	$-\infty$		-2		2		$+\infty$
x^2-4		$+++$	0	$---$	0	$+++$	

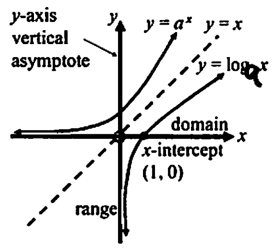
$(-\infty, -2) \cup (2, \infty)$

The Graph of a Logarithmic Function

The graphs of logarithmic functions are similar for different values of a.

$f(x) = \log_a x \ (a > 1)$

Graph of $f(x) = \log_a x \ (a > 1)$
1. domain $(0, \infty)$
2. range $(-\infty, +\infty)$
3. x-intercept $(1, 0)$
4. vertical asymptote $x = 0$ as $x \to 0^+$ $f(x) \to -\infty$
5. increasing
6. continuous
7. one-to-one
8. reflection of $y = a^x$ in $y = x$

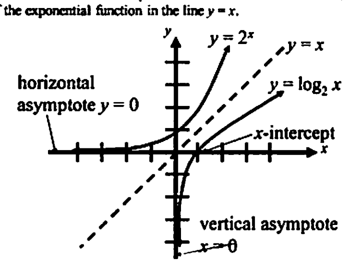


The Graph of Logarithmic Function

Graph $f(x) = \log_2 x$

Since the logarithm function is the *inverse* of the exponential function of the same base, its graph is the reflection of the exponential function in the line $y = x$.

x	2^x
−2	$\frac{1}{4}$
−1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



Find The Domain of Logarithmic Functions

(a) $f(x) = \log_3(x-2)$ (b) $F(x) = \log_2\left(\frac{x+3}{x-1}\right)$

(c) $h(x) = \log_2|x-1|$

$f(x) = \log_3(x-2) \Rightarrow x-2>0$

$x>2 \quad (2,\infty)$

$h(x) = \log_2|x-1| \quad x=1$

$\mathbb{R}-\{1\} \quad |x-1|=|0|\neq 0$

$(-\infty,1)\cup(1,\infty)$ Not accepted.

$g(x)=x^2 \quad \mathbb{R}-\{0\} \quad f(0)=x^2=0^2=0$

$y=\log_a x \quad x>0$

$y=\log_a(x-3) \quad x-3>0$

$x>3 \quad (3,\infty)$

$$F(x) = \log_2\left(\frac{x+3}{x-1}\right)$$

$$\frac{x+3}{x-1} > 0$$

$x+3=0$

$x=-3$

$\downarrow$

zero.

$x-1=0$

$x=1$

$\downarrow$

undefined.

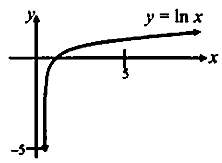
	$-\infty$	-3		1	$+\infty$
$x+3$	$-\ -\ -$	0	$+\ +\ +$	$+$	$+\ +\ +$
$x-1$	$-\ -\ -$	$-$	$-\ -\ -$	/	$+\ +\ +$
$\frac{x-3}{x-1}$	$+\ +\ +$	0	$-\ -\ -\ -$	/	$+\ +\ +\ +$

$$(-\infty,-3)\cup(1,\infty)$$

Natural Logarithmic Function

The function defined by $f(x) = \log_e x = \ln x$
($x > 0$, $e \approx 2.718281\ldots$)
is called the **natural logarithm function.**

$\boxed{y = \ln x \text{ is equivalent to } e^y = x}$



Common Logarithm

The function defined by $f(x) = \log_{10} x = \log x$
where $x > 0$
is called the common logarithm function

$y = \log x \Rightarrow 10^y = x$

Basic Logarithmic Equations

$y = \log_a x$ if and only if $x = a^y$.

Ex:

$$\log_x\left(\frac{1}{8}\right)=3$$

$$x^3=\frac{1}{8}$$

$$x^3=\left(\frac{1}{2}\right)^3$$

$$x=\frac{1}{2}$$

$$y=\log_a x$$

$$a^y=x$$

Ex: $\log_5 x = 3$ $\qquad x \in (0, \infty)$

change in exponential and solve.

$5^3 = x$

$x = 125$

check.

$\log_5 125 = 3$

$\log_5 5^3 = 3$

$3 = 3$ ✓

Ex: $\ln e^{-2x} = 8.$

$e^8 = e^{-2x}$

$\frac{-2x}{-2} = \frac{8}{-2}$

$x = -4$

check.

$\ln e^8 = 8$

$8 = 8.$

$\log_3 243 = 2x+1$

$3^{2x+1} = 243$

$3^{2x+1} = 3^5$

$2x+1=5$
$-1 \quad -1$

$\frac{2x}{2} = \frac{4}{2}$

$x=2$

Evaluate:

$f(x) = \log_3(2x-1)$; $f(5)$

$f(5) = \log_3(2\cdot 5-1)$

$= \log_3 9$

$= \log_3 3^2$

$= 2$